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Generalized Kähler geometry and gerbes

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Abstract: We introduce and study the notion of a biholomorphic gerbe with connection. The biholomorphic gerbe provides a natural geometrical framework for generalized Kähler geometry in a manner analogous to the way a holomorphic line bundle is related to Kähler geometry. The relation between the gerbe and the generalized Kähler potential is discussed.

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Contents

1 Introduction

Recently, generalized Kähler geometry [1] has attracted considerable interest in the physics and mathematics communities. It was discovered in the study of sigma models with $N = (2,2)$ supersymmetry [2, 3], as these have target spaces which necessarily have a generalized Kähler geometry. Such models have proven to be a powerful tool for elucidating further aspects of this geometry. Away from certain loci (irregular points of certain canonical Poisson structures), generalized Kähler geometry can be encoded (locally) in terms of a single real function: the generalized Kähler potential. In the language of supersymmetric sigma models, the generalized Kähler potential is the Lagrangian density in $N = (2,2)$ superspace [2, 4–7]. Being a potential, it is defined modulo certain ambiguities that can be understood both from the geometric and from the sigma model points of view. This paper is an attempt to understand the global issues related to the generalized Kähler potential, and in particular the aspects that can be understood in terms of gerbes.

Gerbes are a geometrical realization of $H^3(M,\mathbb{Z})$ in a manner analogous to the way a line bundle is a geometrical realization of $H^2(M,\mathbb{Z})$. The notion of a holomorphic line bundle is closely related to Kähler geometry. In this paper, we define and investigate the properties of a structure that we call a *biholomorphic gerbe*. A biholomorphic gerbe can be defined on a bicomplex manifold (M, J_+, J_-), i.e., a manifold M equipped with two complex structures. On such a manifold, a biholomorphic gerbe is a collection of those transition functions defined on the triple intersections

$$G_{\alpha\beta\gamma} \; : \; U_\alpha \cap U_\beta \cap U_\gamma \; \rightarrow \; \mathbb{C}^* \,, \tag{1.1}$$

which are biholomorphic,[1] i.e., holomorphic with respect to both complex structures. Moreover, the transition functions are antisymmetric under permutations of the open sets and satisfy a cocycle condition on fourfold intersections. We give a precise definition of biholomorphic gerbes, and show that they arise naturally within generalized Kähler geometry. Where necessary, we shall assume that either M is compact, or that suitable boundary conditions are imposed.

Our analysis is motivated and guided by the sigma model discussion. One objective of the paper is to translate sigma model considerations into proper geometrical terms. To make the paper accessible to both physicists and mathematicians, we review some standard material concerning line bundles, gerbes and supersymmetric sigma models. In all our constructions we adopt the concrete and simple description of gerbes advocated by Hitchin [14, 15]. Note that when we use the terms holomorphic (biholomorphic) functions and their exponentials, we have in mind $\mathcal{O}$, the sheaf of holomorphic (biholomorphic) functions, and $\mathcal{O}^*$, the multiplicative sheaf of nowhere zero holomorphic (biholomorphic) functions, respectively.

Previous uses of gerbes in physics, particularly in the context of WZW models and anti-symmetric tensor gauge fields, have appeared in, e.g., [8–13].

[1] Our use of the word should not be confused with a bijective holomorphic function whose inverse is also holomorphic, which is sometimes also referred to as a biholomorphic function.

The paper is organized as follows. In section 2 we review some basic facts about line bundles and gerbes. Section 3 reviews holomorphic line bundles and their relation to Kähler geometry. Section 4 discusses the notion of a holomorphic gerbe with a Hermitian connection. We point out the appearance of a flat gerbe associated to a Hermitian connection. In section 5 we review the bihermitian description of generalized Kähler geometry and discuss the properties of gerbes with connection associated to this geometry. In particular, we show that generalized Kähler geometry can be encoded in terms of two flat gerbes with additional very special properties. Section 6 is a key part of the present work where we discuss the gluing of the generalized Kähler potential and the relation to biholomorphic gerbes. Section 7 deals with the special case in which the two complex structures commute; then all points are regular and the situation is particularly simple. Section 8 presents a summary of the results as well as some open questions. In the appendix we discuss the example of the natural biholomorphic gerbe with connection on $S^3 \times S^1$.

2 Line bundles and gerbes

In this section we review some standard facts about line bundles with connection and gerbes with connection and introduce our notation. We consider a smooth manifold M with an open cover $\{U_\alpha\}$ where all open sets and intersections are contractible.

2.1 Line bundles and U(1) connections

Let us first recall some facts about line bundles. An S^1-bundle can be thought of as a set of transition functions

$$g_{\alpha\beta} \;:\; U_\alpha \cap U_\beta \;\rightarrow\; S^1\,, \tag{2.1}$$

which satisfy $g_{\alpha\beta} = g^{-1}_{\beta\alpha}$ and the cocycle condition on $U_\alpha \cap U_\beta \cap U_\gamma$

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1\ . \tag{2.2}$$

This condition is trivially satisfied when the transition functions are themselves coboundaries: $g_{\alpha\beta} = h_\alpha h_\beta^{-1}$. S^1-bundles are equivalent to complex line bundles with a Hermitian metric.

To any $\frac{\omega}{2\pi} \in H^2(M, \mathbb{Z})$ we can associate a line bundle with connection as follows. Using the Poincaré lemma, we find 1-forms A_α, functions $\Lambda_{\alpha\beta}$ and constants $d_{\alpha\beta\gamma}$ satisfying

$$\omega = dA_\alpha\,, \qquad A_\alpha \in \Omega^1(U_\alpha)\,, \tag{2.3}$$

$$A_\alpha - A_\beta = d\Lambda_{\alpha\beta}\,, \qquad \Lambda_{\alpha\beta} \in C^\infty(U_\alpha \cap U_\beta)\,, \tag{2.4}$$

$$\Lambda_{\alpha\beta} + \Lambda_{\beta\gamma} + \Lambda_{\gamma\alpha} = d_{\alpha\beta\gamma}\,, \qquad d_{\alpha\beta\gamma} \in 2\pi\ \mathbb{Z} \tag{2.5}$$

where the last relation is guaranteed since $\frac{\omega}{2\pi} \in H^2(M, \mathbb{Z})$ (see e.g. [16] for the proof). Since the coboundary of $\Lambda_{\alpha\beta}$ is equal to 2π times an integer, we can exponentiate it to get transition functions $g_{\alpha\beta} = e^{i\Lambda_{\alpha\beta}}$ that satisfy the cocycle condition (2.2) on triple intersections. The condition (2.4) can be rewritten as

$$iA_\alpha - iA_\beta = g^{-1}_{\alpha\beta} dg_{\alpha\beta} \tag{2.6}$$

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and thus the set of one-forms A_α defines a connection on a line bundle and ω is its curvature. The possible choices of inequivalent connection with the same curvature are parametrized by $H^1(M,\mathbb{R})/H^1(M,\mathbb{Z})$. When the curvature vanishes, this is precisely the space of flat connections, parameterized by their holonomies.

2.2 Gerbes

This definition of a line bundle and a connection can be generalized to gerbes. Gerbes were invented by Giraud [17] and later extensively discussed by Brylinski [18]. We use the simple point of view advocated by Hitchin in [15]. Consider maps defined on each threefold intersection

$$g_{\alpha\beta\gamma} \ : \ U_\alpha \cap U_\beta \cap U_\gamma \ \to \ S^1 \,, \tag{2.7}$$

satisfying

$$g_{\alpha\beta\gamma} = g_{\beta\gamma\alpha} = g_{\gamma\alpha\beta} = g^{-1}_{\beta\alpha\gamma} = g^{-1}_{\alpha\gamma\beta} = g^{-1}_{\gamma\beta\alpha} \tag{2.8}$$

as well as the cocycle condition on $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$

$$g_{\alpha\beta\gamma} g_{\beta\alpha\delta} g_{\gamma\beta\delta} g_{\delta\alpha\gamma} = 1 \ . \tag{2.9}$$

As for the line bundle, this condition is trivially satisfied when the transition functions are themselves coboundaries: $g_{\alpha\beta\gamma} = h_{\alpha\beta} h_{\beta\gamma} h_{\gamma\alpha}$. This data defines a gerbe; we use this definition throughout, though there exist other (equivalent) definitions; for details see [15, 19–21].

2.3 U(1) connections on gerbes

By analogy with the line bundle case we can interpret $\frac{H}{2\pi} \in H^3(M,\mathbb{Z})$ as the curvature[2] of a gerbe with connection. The Poincaré lemma implies the following chain of relations

$$H = dB_\alpha \,, \qquad B_\alpha \in \Omega^2(U_\alpha)\,, \tag{2.10}$$

$$B_\alpha - B_\beta = dA_{\alpha\beta}\,, \qquad A_{\alpha\beta} \in \Omega^1(U_\alpha \cap U_\beta)\,, \tag{2.11}$$

$$A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha} = d\Lambda_{\alpha\beta\gamma}\,, \qquad \Lambda_{\alpha\beta\gamma} \in C^\infty(U_\alpha \cap U_\beta \cap U_\gamma)\,, \tag{2.12}$$

$$\Lambda_{\alpha\beta\gamma} + \Lambda_{\beta\alpha\delta} + \Lambda_{\gamma\beta\delta} + \Lambda_{\delta\alpha\gamma} = d_{\alpha\beta\gamma\delta}\,, \qquad d_{\alpha\beta\gamma\delta} \in 2\pi\ \mathbb{Z}\,, \tag{2.13}$$

where the last one is satisfied as a consequence of $\frac{H}{2\pi} \in H^3(M,\mathbb{Z})$. Using this data we can define the set of functions $g_{\alpha\beta\gamma} \ : \ U_\alpha \cap U_\beta \cap U_\gamma \ \to \ S^1$ given by

$$g_{\alpha\beta\gamma} = e^{i\Lambda_{\alpha\beta\gamma}}\,, \tag{2.14}$$

which, as a result of (2.13), satisfy (2.8)–(2.9) and thus define a gerbe. Equation (2.12) can be rewritten as

$$iA_{\alpha\beta} + iA_{\beta\gamma} + iA_{\gamma\alpha} = g^{-1}_{\alpha\beta\gamma} dg_{\alpha\beta\gamma} \ . \tag{2.15}$$

From (2.14) we see that $\Lambda_{\alpha\beta\gamma}$ are angles,

$$\Lambda_{\alpha\beta\gamma} \in 2\pi\mathbb{R}/\mathbb{Z} \ . \tag{2.16}$$

The above data defines (up to equivalence) a gerbe with connection.

[2]The requirement that the form be integral is an additional requirement on the geometry from the point of view of mathematics. From the physics point of view this requirement is very natural since (on a compact target space) flux has to be quantized to give a well defined quantum theory.

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2.4 Transition line bundle description of gerbes

There is an alternative way to define gerbes [15, 19]. A gerbe can always be made trivial locally. What this means is that in each open set U_α it is possible to choose an open cover $\{V_i^{(\alpha)}\} = U_\alpha \cap U_i$ and functions $h_{ij}^{(\alpha)}$ defined on $V_i^{(\alpha)} \cap V_j^{(\alpha)}$ such that on $V_i^{(\alpha)} \cap V_j^{(\alpha)} \cap V_k^{(\alpha)}$ we have

$$g_{ijk} = h_{ij}^{(\alpha)} h_{jk}^{(\alpha)} h_{ki}^{(\alpha)} . \tag{2.17}$$

The choice of $h_{ij}^{(\alpha)}$ in general is different in each U_α. Only when the gerbe is trivial can one make such a choice globally. In the overlap $U_\alpha \cap U_\beta$ we now have two different trivializations

$$g_{ijk} = h_{ij}^{(\alpha)} h_{jk}^{(\alpha)} h_{ki}^{(\alpha)} = h_{ij}^{(\beta)} h_{jk}^{(\beta)} h_{ki}^{(\beta)} \tag{2.18}$$

Thus $f_{ij} = h_{ij}^{(\alpha)} / h_{ij}^{(\beta)}$ satisfies the cocycle condition which implies that f_{ij} are the transition functions of a line bundle defined on $U_\alpha \cap U_\beta$. This line bundle is called the transition line bundle of the gerbe, and is an equivalent way of encoding the data of the gerbe.

2.5 Flat gerbes

A *flat* gerbe is defined as a gerbe with vanishing curvature: $H = 0$. Then B_α is closed, so that using the Poincaré lemma,

$$\begin{aligned} B_\alpha &= dq_\alpha \,, & (2.19)\\ A_{\alpha\beta} &= q_\alpha - q_\beta + dp_{\alpha\beta} \,, & (2.20)\\ \Lambda_{\alpha\beta\gamma} &= p_{\alpha\beta} + p_{\beta\gamma} + p_{\gamma\alpha} + l_{\alpha\beta\gamma} \,, & (2.21)\\ l_{\alpha\beta\gamma} + l_{\beta\alpha\delta} + l_{\gamma\beta\delta} + l_{\delta\alpha\gamma} &= d_{\alpha\beta\gamma\delta} \,, \qquad d_{\alpha\beta\gamma\delta} \in 2\pi\ \mathbb{Z} \,, & (2.22) \end{aligned}$$

for some q_α which are one-forms on U_α, some $p_{\alpha\beta}$ which are functions on $U_\alpha \cap U_\beta$ and some $l_{\alpha\beta\gamma}$ which are constants. Since the $\Lambda_{\alpha\beta\gamma}$ are angles, the constants $l_{\alpha\beta\delta}$ are also angles, only determined up to an additive factor $2\pi\mathbb{Z}$:

$$l_{\alpha\beta\gamma} \in 2\pi\mathbb{R}/\mathbb{Z} \,. \tag{2.23}$$

Then $\exp(il_{\alpha\beta\gamma})$ is a 2-cocycle so that $l_{\alpha\beta\gamma}/2\pi$ is a 2-cocycle in $\mathbb{R}/\mathbb{Z}$, and represents a Čech class in $H^2(M, \mathbb{R}/\mathbb{Z})$, which corresponds to the holonomy of the flat gerbe. A flat gerbe is then defined by transition functions of the form

$$g_{\alpha\beta\gamma} = e^{i(p_{\alpha\beta}+p_{\beta\gamma}+p_{\gamma\alpha})} e^{il_{\alpha\beta\gamma}} \tag{2.24}$$

which is the product of a trivial piece and a constant.

3 Holomorphic line bundles

Complex holomorphic line bundles over complex manifolds are natural structures in the study of Kähler geometry.

3.1 Holomorphic line bundles and Hermitian connections

When M is a complex manifold, a *holomorphic* line bundle can be defined as a set of holomorphic functions

$$G_{\alpha\beta} \;:\; U_\alpha \cap U_\beta \;\to\; \mathbb{C}^* \,, \tag{3.1}$$

with $G_{\beta\alpha} \equiv (G_{\alpha\beta})^{-1}$ obeying a cocycle condition of the form (2.2) on triple intersections. The topology of the bundle is classified by $H^2(M,\mathbb{Z})$, and a class is represented by a two-form $\frac{\omega}{2\pi} \in H^2(M,\mathbb{Z})$. One can then define an underlying line bundle with connection whose curvature is ω. If furthermore ω is of type $(1,1)$ with respect to the complex structure, (locally) we can write

$$\omega = i\partial\bar{\partial}K_\alpha = \frac{1}{2}dd^cK_\alpha \,, \tag{3.2}$$

where K_α is a real function on U_α, defined up to shifts by the real part of a holomorphic function, and $d^c \equiv i(\bar{\partial} - \partial)$. On $U_\alpha \cap U_\beta$ we have

$$K_\alpha - K_\beta = F_{\alpha\beta}(z) + \bar{F}_{\alpha\beta}(\bar{z}) \,, \tag{3.3}$$

where $F_{\alpha\beta}$ is a holomorphic function on $U_\alpha \cap U_\beta$. On the triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$, eq. (3.3) implies

$$Re\,(F_{\alpha\beta} + F_{\beta\gamma} + F_{\gamma\alpha}) = 0 \;. \tag{3.4}$$

Comparing these relations with the real equations (2.3)–(2.5), we find

$$A_\alpha = \frac{1}{2}d^cK_\alpha \,, \qquad \Lambda_{\alpha\beta} = Im(F_{\alpha\beta}) \;. \tag{3.5}$$

Thus, on triple intersections, the holomorphic function $F_{\alpha\beta}$ satisfies

$$F_{\alpha\beta} + F_{\beta\gamma} + F_{\gamma\alpha} = id_{\alpha\beta\gamma} \in 2\pi i\mathbb{Z} \,, \tag{3.6}$$

which allows us to define the holomorphic transition functions

$$G_{\alpha\beta}(z) = e^{F_{\alpha\beta}(z)} \;:\; U_\alpha \cap U_\beta \;\to\; \mathbb{C}^* \;. \tag{3.7}$$

These transition functions satisfy the standard cocycle condition of the form (2.2) and thus define a holomorphic line bundle. Furthermore, e^K has transition functions $e^{K_\alpha} = G_{\alpha\beta}\bar{G}_{\alpha\beta}e^{K_\beta}$ which are precisely the transition functions that a Hermitian fiber metric should have. Thus e^K is a Hermitian fiber metric and so defines a Hermitian structure on the holomorphic line bundle; such a structure exists whenever $\frac{\omega}{2\pi} \in H^2(M,\mathbb{Z})$ is of type $(1,1)$. The condition satisfied by the transition functions can be rewritten as

$$G_{\alpha\beta}\bar{G}_{\alpha\beta} = e^{K_\alpha}e^{-K_\beta} \;. \tag{3.8}$$

with the right hand side a trivial cocycle, and this is the form of the condition we generalize to gerbes.

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3.2 Relation to Kähler-Hodge geometry

Holomorphic line bundles play an important role in Kähler-Hodge geometry. Consider the Kähler manifold (M, J, g) with $\omega = gJ$ being a Kähler form. Then formula (3.2) provides the local definition of the Kähler potential. The global information about the geometry is encoded in an underlying holomorphic line bundle equipped with a Hermitian metric. When $\omega/2\pi \in H^2(M, \mathbb{Z})$ the manifold is said to be Hodge and the Kähler potential can be given as

$$K_\alpha = \log ||s_\alpha||^2 \,, \tag{3.9}$$

where s_α is a nowhere vanishing section of a holomorphic line bundle and $||s||^2 = hs\bar{s}$ with h the Hermitian metric on this bundle [22].

4 Holomorphic gerbes and connections

For complex manifolds, one can define holomorphic gerbes in complete analogy with holomorphic line bundles.

4.1 Holomorphic gerbes

A *holomorphic* gerbe on a complex manifold M is a set of holomorphic functions

$$G_{\alpha\beta\gamma} \;:\; U_\alpha \cap U_\beta \cap U_\gamma \;\rightarrow\; \mathbb{C}^* \tag{4.1}$$

that are antisymmetric under permutations of the open sets and satisfy a cocycle condition on fourfold intersections. Moreover, if there exist real functions $h_{\alpha\beta}$ on double intersections such that

$$G_{\alpha\beta\gamma}\bar{G}_{\alpha\beta\gamma} = h_{\alpha\beta}h_{\beta\gamma}h_{\gamma\alpha} \tag{4.2}$$

so that $G\bar{G}$ is a trivial cocycle, then we refer to such a gerbe as a holomorphic gerbe with a Hermitian structure, as this is a natural generalisation of the condition (3.8) for hermitian structures on holomorphic line bundles.

4.2 Hermitian connections on holomorphic gerbes

For a closed 3-form H such that $\frac{H}{2\pi} \in H^3(M, \mathbb{Z})$ there exists a gerbe with connection as described in section 2. Assume that H is of type $(2, 1) + (1, 2)$ with respect to the complex structure. Below we explain that this gives a generalization of the holomorphic line bundle which corresponds to a holomorphic gerbe with a Hermitian structure and a connection that respects the Hermitian structure; we refer to this as a holomorphic gerbe with hermitian connection.

On U_α, a connection two-form B_α with $H = dB_\alpha$ can be chosen to be of type $(1, 1)$ (we refer to this as the $(1, 1)$ gauge for B). Then on the double intersection $U_\alpha \cap U_\beta$ we have that $B^{(1,1)}_\alpha - B^{(1,1)}_\beta$ is closed and d^c-closed so that it can be written as $i\bar{\partial}\partial v_{\alpha\beta}$ for some real function $v_{\alpha\beta}$ on $U_\alpha \cap U_\beta$. For later convenience, we write this as the real part of some complex function $\xi_{\alpha\beta}$ on $U_\alpha \cap U_\beta$, $v_{\alpha\beta} = \xi_{\alpha\beta} + \bar{\xi}_{\alpha\beta}$, since the imaginary part plays a role later. Then

$$B^{(1,1)}_\alpha - B^{(1,1)}_\beta = i\bar{\partial}\partial(\xi_{\alpha\beta} + \bar{\xi}_{\alpha\beta}) \,, \tag{4.3}$$

and on the triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$ we find

$$\xi_{\alpha\beta} + \xi_{\beta\gamma} + \xi_{\gamma\alpha} + \bar{\xi}_{\alpha\beta} + \bar{\xi}_{\beta\gamma} + \bar{\xi}_{\gamma\alpha} = -f_{\alpha\beta\gamma}(z) - \bar{f}_{\alpha\beta\gamma}(\bar{z}) \,, \tag{4.4}$$

with $f_{\alpha\beta\gamma}$ a holomorphic function on $U_\alpha \cap U_\beta \cap U_\gamma$. Comparing with (2.10)–(2.12) we find the following relations

$$\begin{aligned} A_{\alpha\beta} &= \frac{i}{2}\partial(\xi_{\alpha\beta} + \bar{\xi}_{\alpha\beta}) - \frac{i}{2}\bar{\partial}(\xi_{\alpha\beta} + \bar{\xi}_{\alpha\beta}) \,, &(4.5)\\ \Lambda_{\alpha\beta\gamma} &= \frac{i}{2}(\bar{f}_{\alpha\beta\gamma} - f_{\alpha\beta\gamma}) \;. &(4.6)\end{aligned}$$

In analogy with the holomorphic line bundle, the imaginary part of the holomorphic function f satisfies (2.13) whereas the real part is a trivial cocycle as a consequence of (4.4). Therefore

$$f_{\alpha\beta\gamma} + f_{\beta\alpha\delta} + f_{\gamma\beta\delta} + f_{\delta\alpha\gamma} \in 2\pi i\, \mathbb{Z} \,, \tag{4.7}$$

and we can define holomorphic transition functions

$$G_{\alpha\beta\gamma} = e^{f_{\alpha\beta\gamma}} \tag{4.8}$$

that satisfy the cocycle condition of the form (2.9) on fourfold intersections. Moreover, due to the property (4.4), the corresponding holomorphic gerbe has a Hermitian structure, i.e., the transition functions satisfy (4.2) with $h_{\alpha\beta} = \exp(-\xi_{\alpha\beta} - \bar{\xi}_{\alpha\beta})$. This is then a holomorphic gerbe with Hermitian connection. The curvature three-form H is necessarily of type $(2,1) + (1,2)$ with respect to the complex structure.

We are particularly interested in generalized (Kähler) geometries for which the closed form H also satisfies $d^c H = 0$, so that $\partial H = \bar{\partial} H = 0$. Locally the $(2,1)$-part of such an H can be written as follows

$$H^{(2,1)} = i\partial\bar{\partial}\lambda_\alpha^{(1,0)} \,, \tag{4.9}$$

where $\lambda_\alpha^{(1,0)}$ is a complex $(1,0)$-form on U_α. Alternatively, in real coordinates we can write

$$H = dd^c(Re\ \lambda_\alpha^{(1,0)}) \;. \tag{4.10}$$

Choosing the $(1,1)$ gauge for B and using (4.9), we obtain

$$B_\alpha^{(1,1)} = i\bar{\partial}\lambda_\alpha^{(1,0)} - i\partial\bar{\lambda}_\alpha^{(0,1)} \;. \tag{4.11}$$

On the double intersection $U_\alpha \cap U_\beta$ we have, using (4.3),

$$\lambda_\alpha^{(1,0)} - \lambda_\beta^{(1,0)} = \partial\xi_{\alpha\beta} + \phi_{\alpha\beta}^{(1,0)} \,, \tag{4.12}$$

where $\xi_{\alpha\beta}$ is a complex function whose real part enters in (4.3) and $\phi^{(1,0)}$ is a holomorphic $(1,0)$-form (i.e., $\bar{\partial}\phi_{\alpha\beta}^{(1,0)} = 0$). This decomposition is not unique: we can always shift a ∂ exact holomorphic one-form between the two terms on the r.h.s. of (4.12), while leaving (4.3) unchanged.

4.3 Transition line bundle on a holomorphic gerbe

Notice that from (4.3) we can interpret $-(\xi + \bar{\xi})$ as the (Kähler) potential for the curvature $\delta B_{\alpha\beta} = B_\alpha - B_\beta$ of the transition line bundle defined on $U_\alpha \cap U_\beta$. This gives an equivalent description of a holomorphic gerbe with Hermitian connection in terms of a holomorphic transition line bundle with hermitian structure (*cf.* the discussion in section 2.4 for the real case).

4.4 Hermitian geometry of (2,0)-supersymmetric sigma models

The Hermitian connection on the holomorphic gerbe described above defines a Hermitian geometry (M, J, g) with complex structure J and Hermitian metric g, which is precisely the geometry of a sigma model with (2,0) supersymmetry [23]. The fundamental 2-form $\omega = gJ$ is of type $(1,1)$ but is not closed. Instead, it satisfies

$$dd^c\omega = 0\,, \tag{4.13}$$

and defines a torsion 3-form

$$H = d^c\omega\,, \tag{4.14}$$

which is closed and d^c-closed and of type $(2,1) + (1,2)$, so that it is given in terms of a 1-form potential $\lambda_\alpha^{(1,0)}$ by (4.10). The fundamental two-form is given in U_α in terms of $\lambda_\alpha^{(1,0)}$ by

$$\omega = -(\bar{\partial}\lambda_\alpha^{(1,0)} + \partial\bar{\lambda}_\alpha^{(0,1)})\ . \tag{4.15}$$

If $\lambda_\alpha^{(1,0)} = -i\partial K_\alpha$, then the manifold is Kähler with Kähler potential K_α and $H = 0$.

4.5 An associated flat gerbe

Because the gerbe curvature (4.10) has 1-form potential $Re(\lambda_\alpha^{(1,0)})$, we can define a collection of locally defined closed forms

$$\mathcal{F}_\alpha = d\ Re(\lambda_\alpha^{(1,0)})\ . \tag{4.16}$$

We interpret this collection of two-forms as connection two-forms for a flat gerbe. We elaborate on this flat gerbe in the following sections.

5 Generalized Kähler geometry

In this section we review the definition of generalized Kähler geometry and discuss the gerbe associated with this geometry.

5.1 Bihermitian formulation of generalized Kähler geometry

A generalized Kähler manifold (M, J_+, J_-, g) is a manifold M with two complex structures $J_\pm$ and a bihermitian metric g satisfying the integrability conditions

$$d^c_+\omega_+ + d^c_-\omega_- = 0\,, \qquad dd^c_\pm\omega_\pm = 0\,, \tag{5.1}$$

where $\omega_\pm = gJ_\pm$ and $d^c_\pm$ are the $i(\bar\partial - \partial)$ operators associated with the complex structures $J_\pm$. The conditions (5.1) imply that we can define a closed three form

$$H = d^c_+\omega_+ = -d^c_-\omega_- \,, \quad dH = 0 \tag{5.2}$$

which is also d^c-closed, $d^c H = 0$.

5.2 Gerbes on a generalized Kähler geometry

If $\frac{H}{2\pi} \in H^3(M, \mathbb{Z})$ then we have a gerbe with connection. The condition $\frac{H}{2\pi} \in H^3(M, \mathbb{Z})$ is necessary for the sigma model with Wess-Zumino term specified by H to give a well-defined quantum theory (on a compact target space), and we assume that this holds. The definitions $\omega_\pm = gJ_\pm$ and (5.2) imply that H is a $(2,1) + (1,2)$ form with respect to *both* complex structures:

$$H = H_+^{(2,1)} + H_+^{(1,2)} = H_-^{(2,1)} + H_-^{(1,2)} \,, \tag{5.3}$$

$$H_\pm^{(2,1)} - H_\pm^{(1,2)} = \mp i d\omega_\pm \,, \tag{5.4}$$

which implies

$$H_+^{(2,1)} - H_-^{(2,1)} = H_-^{(1,2)} - H_+^{(1,2)} = -\frac{i}{2} d\left(\omega_+ + \omega_-\right) , \tag{5.5}$$

$$H_+^{(2,1)} - H_-^{(1,2)} = H_-^{(2,1)} - H_+^{(1,2)} = -\frac{i}{2} d\left(\omega_+ - \omega_-\right) . \tag{5.6}$$

5.3 Two gerbe connections

As H is a $(2,1) + (1,2)$ form with respect to both complex structures, the discussion from the previous section about holomorphic hermitian gerbes applies twofold: the globally defined curvature 3-form H is the curvature of two holomorphic gerbes with Hermitian connection, one associated with each of the two complex structures, $J_\pm$. We use the same notation as in section 4, adding the subscript $\pm$ to indicate the relevant complex structure. For a given H we can choose a connection $B_{+\alpha}$ that is $(1,1)$ respect to J_+ or a connection $B_{-\alpha}$ which is $(1,1)$ respect to J_-, with $H = dB_{\pm\alpha}$. Note that in general these specify inequivalent gerbes with connection.

For each choice of connection $B_\pm$ there are descendants $\lambda_\pm^{(1,0)}, \xi_\pm, f_\pm$ satisfying formulas of the form (4.11)–(4.6) with $\pm$ added appropriately to indicate the choice of complex structure.

5.4 A flat gerbe

Two gerbes with connection associated to the same curvature three-form differ by a flat gerbe, so $H = dB_{+\alpha}^{(1,1)} = dB_{-\alpha}^{(1,1)}$ and hence $d[B_{+\alpha}^{(1,1)} - B_{-\alpha}^{(1,1)}] = 0$. Then we have(*cf.* (2.19)–(2.21))

$$B_{+\alpha}^{(1,1)} - B_{-\alpha}^{(1,1)} = dq_\alpha \,, \tag{5.7}$$

$$A_{\alpha\beta}^+ - A_{\alpha\beta}^- = q_\alpha - q_\beta + dp_{\alpha\beta} \,, \tag{5.8}$$

$$\Lambda_{\alpha\beta\gamma}^+ - \Lambda_{\alpha\beta\gamma}^- = p_{\alpha\beta} + p_{\beta\gamma} + p_{\gamma\alpha} + l_{\alpha\beta\gamma} \,, \tag{5.9}$$

where q_α are one-forms on U_α, $p_{\alpha\beta}$ are functions on $U_\alpha \cap U_\beta$ and $l_{\alpha\beta\gamma}$ are constants. Using (4.4) and the definition (4.8) we conclude that the transition functions

$$g_{\alpha\beta\gamma} = G^+_{\alpha\beta\gamma}\bar{G}^-_{\alpha\beta\gamma} = e^{i(\Lambda^+_{\alpha\beta\gamma} - \Lambda^-_{\alpha\beta\gamma})} \tag{5.10}$$

correspond to a flat gerbe, as they are of the form (2.24).

However, generalized Kähler geometry contains more structure than two Hermitian gerbes with Hermitian connections that have the same curvature. In particular, (5.2) implies H is $d^c_\pm$-exact. The $\omega_\pm$ are $(1,1)$ forms and they have nice expressions in terms of one-form potentials $\lambda^{(1,0)}_\pm$ (see the previous section)

$$\omega^{(1,1)}_\pm = \mp\left(\partial_\pm \bar{\lambda}^{(0,1)}_\pm + \bar{\partial}_\pm \lambda^{(1,0)}_\pm\right) . \tag{5.11}$$

Since $\omega_\pm$ are globally defined two forms we can conclude from (4.12) and (5.11) that in $U_\alpha \cap U_\beta$

$$dd^c_\pm\left(\bar{\xi}_\pm - \xi_\pm\right) = 0\,,$$

which tells us that the imaginary part of $(\xi_{\alpha\beta})_\pm$ can be written as the imaginary part of a holomorphic function for both complex structures. This in turn tells us that $\partial_\pm Im\xi_\pm$ is a holomorphic one-form and can therefore be absorbed in $\phi_\pm$ in equation (4.12). Thus we conclude that ξ can always be chosen to be real. With a real ξ, (4.12) and (4.4) imply that

$$\phi^{(1,0)}_{\alpha\beta} + \phi^{(1,0)}_{\beta\gamma} + \phi^{(1,0)}_{\gamma\alpha} = \frac{1}{2}\partial f_{\alpha\beta\gamma} = \frac{1}{2}G^{-1}_{\alpha\beta\gamma}dG_{\alpha\beta\gamma}\,, \tag{5.12}$$

It is possible to choose a two-form connection B of type $(2,0)+(0,2)$ so that $H^{(2,1)} = dB^{(2,0)}_\alpha$.[3] The reality of ξ allows us to choose an $A^{(1,0)}_{\alpha\beta}$ with the following chain of relations

$$H^{(2,1)} = dB^{(2,0)}_\alpha\,, \tag{5.13}$$

$$B^{(2,0)}_\alpha - B^{(2,0)}_\beta = dA^{(1,0)}_{\alpha\beta}\,, \tag{5.14}$$

$$A^{(1,0)}_{\alpha\beta} + A^{(1,0)}_{\beta\gamma} + A^{(1,0)}_{\gamma\alpha} = -\frac{i}{2}G^{-1}_{\alpha\beta\gamma}dG_{\alpha\beta\gamma}\,, \tag{5.15}$$

where we have suppressed the labels $\pm$ denoting the complex structure. We stress that these relations do *not* hold in general for holomorphic gerbes with Hermitian connection.

These relations imply that

$$\partial B^{(2,0)}_\alpha = 0, \qquad \bar{\partial}A^{(1,0)}_{\alpha\beta} = 0\ . \tag{5.16}$$

From (4.9), we see that $B^{(2,0)}_\alpha$ can be chosen to be

$$B^{(2,0)}_\alpha = -i\partial\lambda^{(1,0)}_\alpha \tag{5.17}$$

so that (4.12) implies that a holomprphic $(1,0)$ form satisfying (5.14) is

$$A^{(1,0)}_{\alpha\beta} = -i\phi^{(1,0)}_{\alpha\beta}\ . \tag{5.18}$$

Then (5.15) follows from (5.12).

[3] We refer to this as the $(2,0)$ gauge. See, e.g., [24] for this and other gauges.

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5.5 Two locally defined symplectic forms

Combining the relations (5.2) and (5.13) we are led to a very unexpected description of generalized Kähler geometry in terms of locally defined closed nondegenerate forms, which can also be thought of in a more global language as flat gerbe connections. Using the relations (4.15) and (5.17), the two-forms (4.16) can be written as

$$\mathcal{F}^{\pm}_{\alpha} = \frac{i}{2}\left(B^{(2,0)}_{\pm\alpha} - B^{(0,2)}_{\pm\alpha}\right) \mp \frac{1}{2}\omega_{\pm} \ . \tag{5.19}$$

This is a collection of closed two-forms $\mathcal{F}^{\pm}_{\alpha}$ defined on U_α; $d\mathcal{F}^{\pm}_{\alpha} = 0$ follows immediately from (5.2) and (5.13). Moreover, these forms are nondegenerate, i.e., $\mathcal{F}^{\pm}_{\alpha}$ are symplectic structures on U_α. These forms (and linear combinations of them) can be interpreted as connection forms for a flat gerbe, which may or may not be trivial. It has been shown that the flat gerbes specified by the collection of two forms

$$\frac{1}{2}\left(\mathcal{F}^{+}_{\alpha} \pm \mathcal{F}^{-}_{\alpha}\right) ,$$

plays an essential role in the definition of topological string theory on generalized Kähler manifolds [24]. The different choices of sign correspond to either A- or B-twist topological models.

5.6 Another characterization of generalized Kähler geometry

The forms $\mathcal{F}^{\pm}_{\alpha}$ encode the full local geometrical data of the generalized Kähler geometry on a bicomplex manifold. We have the following theorem holding locally:

Theorem. *Consider a coordinate patch U of a bicomplex manifold. Suppose there exist two symplectic forms $\mathcal{F}^{\pm}$ that tame the complex structures $J_{\pm}$ respectively, i.e., for any nonzero tangent vector v we have*

$$\mathcal{F}^{\pm}(v, J_{\pm}v) > 0 \ .$$

If in addition

$$\mathcal{F}^{+}J_{+} - J^{t}_{-}\mathcal{F}^{-}$$

is a closed two-form, then this data defines a generalized Kähler geometry on U.

Proof: The proof is straightforward. Note that the condition in the theorem is that $\mathcal{F}^{+}(v, J_{+}v) > 0$ and $\mathcal{F}^{-}(v, J_{-}v) > 0$. Let us decompose the 2-form $\mathcal{F}^{+}$ with respect to the complex structures J_{+} and $\mathcal{F}^{-}$ with respect to J_{-},

$$\mathcal{F}^{\pm} = (\mathcal{F}^{\pm})^{(2,0)} + (\mathcal{F}^{\pm})^{(0,2)} + (\mathcal{F}^{\pm})^{(1,1)} \ . \tag{5.20}$$

Comparing with (5.19), we identify

$$(\mathcal{F}^{\pm})^{(1,1)} = \mp\frac{1}{2}\omega_{\pm} \, , \qquad (\mathcal{F}^{\pm})^{(2,0)} = \frac{i}{2}(B^{\pm})^{(2,0)} \ . \tag{5.21}$$

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Using these identifications we decompose $\mathcal{F}^+ J_+$ and $J_-^t \mathcal{F}^-$ into symmetric and anti-symmetric parts,

$$\mathcal{F}^+ J_+ = -\frac{1}{2}B^+ + \frac{1}{2}g_+ \,, \qquad J_-^t \mathcal{F}^- = -\frac{1}{2}B^- + \frac{1}{2}g_- \,, \tag{5.22}$$

where $g_\pm$ are symmetric tensors and $B^\pm$ are anti-symmetric tensors. If the corresponding symplectic structures tame the complex structures $J_\pm$, then $g_\pm$ are positive-definite and so define metrics. The second condition in the theorem implies that

$$g_+ = g_- \equiv g \,, \qquad dB^+ = dB^- \equiv H \ . \tag{5.23}$$

Then the condition $d\mathcal{F}^\pm = 0$ implies

$$dB^+ = d_+^c \omega_+ = dB^- = -d_-^c \omega_- \,, \tag{5.24}$$

with $\omega_\pm = gJ_\pm$. Thus we obtain the standard local bihermitian formulation of generalized Kähler geometry.

This theorem can be understood as an alternative local description of generalized Kähler geometry; it is formulated entirely in terms of locally *bisymplectic* bicomplex geometry. In order to make the theorem global one has to specify the way the forms $\mathcal{F}^\pm$ patch together in overlaps. As should be clear from the previous discussion, they have the transition functions of flat gerbe connections. A special case of this theorem in which $\mathcal{F}^\pm$ are globally defined and obey $\mathcal{F}^+ = -\mathcal{F}^-$ was considered in [25].

In this section we have presented a number of results concerning the gerbe structures of generalized Kähler geometry. However using the notion of the generalized Kähler potential we can analyse the underlying structures further.

6 The generalized Kähler potential

In [2], it was found that the $N = (2, 2)$ superspace Lagrangian of supersymmetric sigma models encoded particular examples of generalized Kähler geometry in terms of a single generalized Kähler potential; this was extended to more generic situations in [4], and conjectured to hold generally in [5]. In [6], it was proved that this is indeed the case: away from irregular points, a generalized Kähler manifold can be locally described in terms of a single real function K, the generalized Kähler potential. In this section we discuss the gluing properties of the generalized Kähler potential and their relation to the underlying gerbe with connection.

6.1 Review of the potential

We now briefly review the local geometry of a generalized Kähler manifold and its description in terms of a potential. Generalized Kähler manifolds have a rich underlying Poisson geometry [26, 27]: there are two real Poisson structures $\pi_\pm = (J_+ \pm J_-)g^{-1}$. We call a point regular if there exists a neighborhood of that point where $\pi_\pm$ have constant rank. In addition there is a third Poisson structure $\sigma = [J_+, J_-]g^{-1} = \pi_- g \pi_+$ which can be written as the real part of a Poisson structure that is holomorphic with respect to either complex structure: $\sigma = \frac{1}{2}[\sigma_\pm^{(2,0)} + \sigma_\pm^{(0,2)}]$. In the neighborhood of a regular point we can introduce coordinates adapted to the corresponding symplectic foliations: we introduce coordinates $(\phi, \bar\phi, \chi, \bar\chi, X_L, \bar X_L, X_R, \bar X_R)$ such that $(d\phi, d\bar\phi)$ span the kernel of π_- and $(d\chi, d\bar\chi)$ span the kernel of π_+. The remaining coordinates $X_{L,R}, \bar X_{L,R}$ lie along the leaves of σ. We choose Darboux coordinates X_L, Y_L for $\sigma_+^{(2,0)} = dX_L \wedge dY_L$ and X_R, Y_R for $\sigma_-^{(2,0)} = dX_R \wedge dY_R$; then a polarization is just a choice of an equal number of X_L and X_R coordinates out of the set $\{X_L, Y_L, X_R, Y_R\}$. The generalized Kähler potential is locally a function of $(\phi, \bar\phi, \chi, \bar\chi, X_L, \bar X_L, X_R, \bar X_R)$, and all geometrical quantities are given in terms of second derivatives of K. The relations are linear if $\sigma = 0$ (see the next section) but are nonlinear in general. For further details the reader can consult [6, 7, 28].

The generalized Kähler potential K is not defined uniquely by the geometry; the precise form of the relation between K and the geometry depends on the choice of polarization on the leaves of σ, and for a given polarization K can be shifted by generalized Kähler gauge transformations without changing the geometry, as is described below. A change of a polarization on a leaf of σ corresponds to a coordinate change (symplectomorphism) and transforms K by a Legendre transformation.

We now focus on generalized Kähler manifolds that are regular everywhere; the case with irregular points is more subtle and we leave it to future investigations.

6.2 Generalized Kähler transformations on overlaps

When all the Poisson structures are regular everywhere on the manifold, we can choose natural coordinates in each patch and in their intersections. In addition, K can transform when we move between different patches. For instance, if we have the same polarization in U_α and U_β, then requiring that K_α and K_β define the same geometry on $U_\alpha \cap U_\beta$ implies

$$K_\alpha - K_\beta = F^+_{\alpha\beta}(\phi, \chi, X_L) + \bar F^+_{\alpha\beta}(\bar\phi, \bar\chi, \bar X_L) + F^-_{\alpha\beta}(\phi, \bar\chi, X_R) + \bar F^-_{\alpha\beta}(\bar\phi, \chi, \bar X_R)\,, \tag{6.1}$$

for some special J_+-holomorphic function $F^+_{\alpha\beta}$ and special J_--holomorphic function $F^-_{\alpha\beta}$.[4] These functions are defined in turn up to the following shifts

$$F^+_{\alpha\beta}(\phi, \chi, X_L) \to F^+_{\alpha\beta}(\phi, \chi, X_L) + \rho_{\alpha\beta}(\phi) + \sigma_{\alpha\beta}(\chi)\,, \tag{6.2}$$
$$\bar F^+_{\alpha\beta}(\bar\phi, \bar\chi, \bar X_L) \to \bar F^+_{\alpha\beta}(\bar\phi, \bar\chi, \bar X_L) + \bar\rho_{\alpha\beta}(\bar\phi) + \bar\sigma_{\alpha\beta}(\bar\chi)\,, \tag{6.3}$$
$$F^-_{\alpha\beta}(\phi, \bar\chi, X_R) \to F^-_{\alpha\beta}(\phi, \bar\chi, X_R) - \rho_{\alpha\beta}(\phi) - \bar\sigma_{\alpha\beta}(\bar\chi)\,, \tag{6.4}$$
$$\bar F^-_{\alpha\beta}(\bar\phi, \chi, \bar X_R) \to \bar F^-_{\alpha\beta}(\bar\phi, \chi, \bar X_R) - \bar\rho_{\alpha\beta}(\bar\phi) - \sigma_{\alpha\beta}(\chi)\,. \tag{6.5}$$

Here $\rho_{\alpha\beta}$ is holomorphic with respect to both $J_\pm$, and hence is *biholomorphic*, and $\sigma_{\alpha\beta}$ is holomorphic with respect to J_+ and $-J_-$ and hence is *twisted biholomorphic* (see the discussion below in subsection 6.5).

From (6.1) we have the following relation

$$\mathrm{Re}(F^+_{\alpha\beta} + F^+_{\beta\alpha} + F^-_{\alpha\beta} + F^-_{\beta\alpha}) = 0\,; \tag{6.6}$$

[4] An, e.g., J_+-holomorphic function could depend on (ϕ, χ, X_L, Y_L); a *special* J_+-holomorphic function is defined with respect to a choice of polarization and depends only on (ϕ, χ, X_L).

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using the ambiguities (6.2)–(6.5) we can choose F^+'s and F^-'s to be separately antisymmetric under interchange of the open sets:

$$F^+_{\alpha\beta} = -F^+_{\beta\alpha}\,, \qquad F^-_{\alpha\beta} = -F^-_{\beta\alpha}\,. \tag{6.7}$$

6.3 Biholomorphic and twisted biholomorphic cocycles on triple overlaps

Taking the coboundary of (6.1) we get on $U_\alpha \cap U_\beta \cap U_\gamma$

$$\mathrm{Re}(F^+_{\alpha\beta} + F^+_{\beta\gamma} + F^+_{\gamma\alpha} + F^-_{\alpha\beta} + F^-_{\beta\gamma} + F^-_{\gamma\alpha}) = 0\,, \tag{6.8}$$

which implies that the coboundary of $F^\pm$ can be expressed in terms of biholomorphic and twisted biholomorphic functions $c_{\alpha\beta\gamma}(\phi)$ and $b_{\alpha\beta\gamma}(\chi)$:

$$\begin{aligned}
F^+_{\alpha\beta}(\phi,\chi,X_L) + F^+_{\beta\gamma}(\phi,\chi,X_L) + F^+_{\gamma\alpha}(\phi,\chi,X_L) &= i\left(c_{\alpha\beta\gamma}(\phi) - b_{\alpha\beta\gamma}(\chi)\right)\,, && (6.9)\\
\bar{F}^+_{\alpha\beta}(\bar\phi,\bar\chi,\bar X_L) + \bar{F}^+_{\beta\gamma}(\bar\phi,\bar\chi,\bar X_L) + \bar{F}^+_{\gamma\alpha}(\bar\phi,\bar\chi,\bar X_L) &= -i\left(\bar c_{\alpha\beta\gamma}(\bar\phi) - \bar b_{\alpha\beta\gamma}(\bar\chi)\right)\,, && (6.10)\\
F^-_{\alpha\beta}(\phi,\bar\chi,X_R) + F^-_{\beta\gamma}(\phi,\bar\chi,X_R) + F^-_{\gamma\alpha}(\phi,\bar\chi,X_R) &= -i\left(c_{\alpha\beta\gamma}(\phi) + \bar b_{\alpha\beta\gamma}(\bar\chi)\right)\,, && (6.11)\\
\bar{F}^-_{\alpha\beta}(\bar\phi,\chi,\bar X_R) + \bar{F}^-_{\beta\gamma}(\bar\phi,\chi,\bar X_R) + \bar{F}^-_{\gamma\alpha}(\bar\phi,\chi,\bar X_R) &= i\left(\bar c_{\alpha\beta\gamma}(\bar\phi) + b_{\alpha\beta\gamma}(\chi)\right)\,, && (6.12)
\end{aligned}$$

where the c and b functions are defined up to constant shifts

$$\begin{aligned}
c_{\alpha\beta\gamma}(\phi) &\to c_{\alpha\beta\gamma}(\phi) + h_{\alpha\beta\gamma}\,, && (6.13)\\
b_{\alpha\beta\gamma}(\chi) &\to b_{\alpha\beta\gamma}(\chi) + h_{\alpha\beta\gamma}\,. && (6.14)
\end{aligned}$$

At the same time, using (6.7) we can derive the relations

$$\begin{aligned}
c_{\alpha\beta\gamma}(\phi) + c_{\beta\alpha\gamma}(\phi) - b_{\alpha\beta\gamma}(\chi) - b_{\beta\alpha\gamma}(\chi) &= 0\,, && (6.15)\\
c_{\alpha\beta\gamma}(\phi) + c_{\beta\alpha\gamma}(\phi) + \bar b_{\alpha\beta\gamma}(\bar\chi) + \bar b_{\beta\alpha\gamma}(\bar\chi) &= 0\,. && (6.16)
\end{aligned}$$

Using the ambiguities (6.13)–(6.14) and the relations (6.15)–(6.16) we can always choose such c and b which satisfy

$$c_{\alpha\beta\gamma} = -c_{\beta\alpha\gamma} = -c_{\alpha\gamma\beta} = -c_{\gamma\beta\alpha}\,, \qquad b_{\alpha\beta\gamma} = -b_{\beta\alpha\gamma} = -b_{\alpha\gamma\beta} = -b_{\gamma\beta\alpha}\,. \tag{6.17}$$

6.4 Integral cocycles on four-fold overlaps

Now on $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$ we have

$$\begin{aligned}
c_{\alpha\beta\gamma} + c_{\beta\alpha\delta} + c_{\gamma\beta\delta} + c_{\delta\alpha\gamma} - b_{\alpha\beta\gamma} - b_{\beta\alpha\delta} - b_{\gamma\beta\delta} - b_{\delta\alpha\gamma} &= 0\,, && (6.18)\\
c_{\alpha\beta\gamma} + c_{\beta\alpha\delta} + c_{\gamma\beta\delta} + c_{\delta\alpha\gamma} + \bar b_{\alpha\beta\gamma} + \bar b_{\beta\alpha\delta} + \bar b_{\gamma\beta\delta} + \bar b_{\delta\alpha\gamma} &= 0\,. && (6.19)
\end{aligned}$$

These conditions imply

$$\begin{aligned}
c_{\beta\gamma\delta} + c_{\delta\gamma\alpha} + c_{\alpha\beta\delta} + c_{\beta\alpha\gamma} &= \frac{i}{4} d_{\alpha\beta\gamma\delta}\,, && (6.20)\\
b_{\beta\gamma\delta} + b_{\delta\gamma\alpha} + b_{\alpha\beta\delta} + b_{\beta\alpha\gamma} &= \frac{i}{4} d_{\alpha\beta\gamma\delta}\,. && (6.21)
\end{aligned}$$

In particular using the formulas in [6] for H in terms of K we see that if $\frac{H}{2\pi} \in H^3(M,\mathbb{Z})$ then $d_{\alpha\beta\gamma\delta} \in 2\pi\mathbb{Z}$.

Let us elaborate on the relation between the generalized Kähler potential and the description of a gerbe with connection from section 4. The key observation is that using formulas from [6] we can find a *linear* expression for the one-form potentials $\lambda_\pm$ (see (5.11)) in terms of the generalized Kähler potential as

$$\begin{aligned}
\lambda_+^{(1,0)} + \bar{\lambda}_+^{(0,1)} &= -i\left(\frac{\partial K}{\partial X_R^{\alpha'}} dX_R^{\alpha'} + \frac{\partial K}{\partial \phi^a} d\phi^a - \frac{\partial K}{\partial \chi^{a'}} d\chi^{a'}\right) - c.c. \qquad (6.22)\\
&= -\left(\frac{\partial K}{\partial X_R^{\mathcal{A}'}} (J_-)^{\mathcal{A}'}{}_{\mathcal{B}'} dX_R^{\mathcal{B}'} + \frac{\partial K}{\partial \phi^A} (J_-)^A{}_B d\phi^B + \frac{\partial K}{\partial \chi^{A'}} (J_-)^{A'}{}_{B'} d\chi^{B'}\right),\\
\lambda_-^{(1,0)} + \bar{\lambda}_-^{(0,1)} &= i\left(\frac{\partial K}{\partial X_L^{\alpha}} dX_L^{\alpha} + \frac{\partial K}{\partial \phi^a} d\phi^a + \frac{\partial K}{\partial \chi^{a'}} d\chi^{a'}\right) - c.c.\\
&= \left(\frac{\partial K}{\partial X_L^{\mathcal{A}}} (J_+)^{\mathcal{A}}{}_{\mathcal{B}} dX_L^{\mathcal{B}} + \frac{\partial K}{\partial \phi^A} (J_+)^A{}_B d\phi^B + \frac{\partial K}{\partial \chi^{A'}} (J_+)^{A'}{}_{B'} d\chi^{B'}\right), \qquad (6.23)
\end{aligned}$$

(see [6] for our index conventions.) Then comparing with (4.12), we find that in the double overlap $U_\alpha \cap U_\beta$

$$\xi_+ = i(\bar{F}^- - F^-) \qquad (6.24)$$

$$\phi_+^{(1,0)} = i\left(\frac{\partial F^+}{\partial \chi^{a'}} d\chi^{a'} - \frac{\partial F^+}{\partial \phi^a} d\phi^a\right) \qquad (6.25)$$

$$\xi_- = i(F^+ - \bar{F}^+) \qquad (6.26)$$

$$\phi_-^{(1,0)} = i\left(\frac{\partial F^-}{\partial \phi^a} d\phi^a - \frac{\partial F^-}{\partial \bar{\chi}^{\bar{a}'}} d\bar{\chi}^{\bar{a}'}\right) \qquad (6.27)$$

from which follows (using (4.6) and (6.9)–(6.12))

$$\Lambda_{\alpha\beta\gamma} = i\left(\bar{c}_{\alpha\beta\gamma}(\bar{\phi}) - c_{\alpha\beta\gamma}(\phi) + \bar{b}_{\alpha\beta\gamma}(\bar{\chi}) - b_{\alpha\beta\gamma}(\chi)\right). \qquad (6.28)$$

Assuming the integrality of H and the relations (6.20), (6.21) we find that (2.13) is satisfied with $d_{\alpha\beta\gamma\delta} \in 2\pi\mathbb{Z}$. This allows us to introduce the functions

$$G_{\alpha\beta\gamma}(\phi) = e^{4c_{\alpha\beta\gamma}(\phi)}, \qquad F_{\alpha\beta\gamma}(\chi) = e^{4b_{\alpha\beta\gamma}(\chi)}, \qquad (6.29)$$

defined over triple intersections

$$G_{\alpha\beta\gamma}, F_{\alpha\beta\gamma} \;:\; U_\alpha \cap U_\beta \cap U_\gamma, \rightarrow\; \mathbb{C}^*, \qquad (6.30)$$

which are antisymmetric under permutations of the open sets and satisfy the cocycle condition on the four-fold intersection.

6.5 Comments on biholomorphic functions

Note that G depends only on ϕ and F depends only on χ. On any manifold M with two complex structures J_+ and J_-, a complex function f is biholomorphic if it is holomorphic with respect to both complex structures, i.e.,

$$(d - i d^c_\pm) f = 0\,. \qquad (6.31)$$

As a result of this f also satisfies

$$(d_+^c - d_-^c)f = 0 \ . \tag{6.32}$$

Thus on a generalized Kähler manifold a biholomorphic function f is a Casimir function of the Poisson structure π_-, i.e., $df \in \ker \pi_-$. In the coordinate system we use, this means that a biholomorphic function is a function of only ϕ. Analogously one can consider functions which are holomorphic with respect to J_+ and antiholomorphic with respect to J_-, which we call twisted biholomorphic. Such functions are the Casimir functions for π_+ and in our coordinates depend only on χ.

6.6 Biholomorphic gerbes

Combining the above, we arrive at the definition of a biholomorphic gerbe on a bicomplex manifold (M, J_+, J_-) as a collection of biholomorphic functions $G_{\alpha\beta\gamma}$ that are antisymmetric under permutations of the open sets and satisfy the cocycle condition on four-fold intersections. Analogously we define a twisted biholomorphic gerbe as a collection of twisted biholomorphic functions $F_{\alpha\beta\gamma}$. In the analysis above we have shown that a bihomolomorphic gerbe and a twisted bihomolomorphic gerbe naturally appear in the context of generalized Kähler geometry through the discussion of gluing of the generalized Kähler potential. Moreover, by exponentiating the equations (6.9)–(6.12) we arrive at the following relation between biholomorphic and twisted biholomorphic transition functions

$$G_{\alpha\beta\gamma}F^{-1}_{\alpha\beta\gamma} = h^+_{\alpha\beta}h^+_{\beta\gamma}h^+_{\gamma\alpha} \,, \qquad\qquad G_{\alpha\beta\gamma}\bar{F}_{\alpha\beta\gamma} = h^-_{\alpha\beta}h^-_{\beta\gamma}h^-_{\gamma\alpha} \,, \tag{6.33}$$

where $h^\pm_{\alpha\beta} = exp(\mp 4iF^\pm_{\alpha\beta})$ are $J_\pm$-holomorphic functions of special form.

Alternatively, one can define this structure using transition line bundles; essentially, one defines a holomorphic transition line bundle for each complex structure on the generalized Kähler manifold M and imposes a compatibility condition which is equivalent to (6.33). This description is sometimes more compact, and is used in the example of a biholomorphic gerbe on $S^1 \times S^3$ given in the appendix.

In the special case in which $(J_+ - J_-)$ is invertible everywhere on M, then the only biholomorphic functions are constants. Indeed in this case H is cohomologically trivial.[5] Analogously one can show that if $(J_+ + J_-)$ is invertible everywhere on M, then the only twisted biholomorphic functions are constants and again H is exact.

The present analysis is not complete. We have focused on the linear transformations (6.1) of K; however, as shown in [6], K also encodes a choice of polarization for symplectic leaves on the manifold. A change of this polarization is realized by a Legendre transformation, and we have not explored how this nonlinear transformation intertwines with the linear transitions discussed above. We leave the problem of finding a full geometrical interpretation of K for future research.

[5] This is easy to see in the formulation of generalized Kähler geometry in terms of generalized complex structures. In this case one of the generalized complex structures is of symplectic type and therefore H is exact.

7 Bihermitian local product spaces

In this section we consider the special case of regular generalized Kähler manifolds for which $\sigma = [J_+, J_-]g^{-1} = 0$. This means that we can simply exclude $X_{L,R}$ and $\bar{X}_{L,R}$ from all formulas of the previous section. This case is considerably simpler than the general case since all geometrical objects depend linearly on the generalized Kähler potential and it is guaranteed that every point is regular. Such a space carries a local product structure $\Pi = J_+ J_-$ and is called a Bihermitian Local Product space (or BiLP for short). Locally, a BiLP looks like a product of two Kähler manifolds and in a way H is responsible for making this product nontrivial. The simplest compact example of such geometry is $S^3 \times S^1$, and this example is analyzed in the appendix.

7.1 The generalized Kähler potential on a BiLP

On a BiLP, the complex structures $J_\pm$ commute, and hence the differentials $d^c_\pm$ obey

$$d^c_+ d^c_- = -d^c_- d^c_+ \tag{7.1}$$

as well as $dd^c_\pm = -d^c_\pm d$. Hence, on a BiLP we can write the closed three-form H as

$$H = \frac{1}{2} dd^c_+ d^c_- K_\alpha \,, \tag{7.2}$$

where K_α is a real function on a patch U_α and H is $(2,1)+(1,2)$ form with respect to both complex structures $J_\pm$. Comparing (7.2) and (5.2), we see that the two-forms $\omega_\pm$ can also be simply expressed in terms of K_α.

There are a number of compatible distributions on TM given by J_+, J_- and Π. This allows us to have quadruple-grading on the differential forms and split the de Rham differential as follows

$$d = \partial_\phi + \partial_\chi + \partial_{\bar{\phi}} + \partial_{\bar{\chi}} \,. \tag{7.3}$$

Further details on the geometry and notation can be found in [2, 28].

7.2 Gerbes on a BiLP

It is a straightforward exercise to work out the whole chain of the relations (4.3)–(4.12) and (5.13)–(5.15) in terms of the data coming from the gluing of the generalized Kähler potential K_α. As particular examples, let us present the following expressions (also see the equations (6.22)–(6.27) from the previous section). Choose, e.g., J_+ as the complex structure with respect to which the differential forms are graded; the relations (5.13)–(5.15) are satisfied by

$$B^{(2,0)}_\alpha = 2\left(\partial_{\bar{\phi}}\partial_{\bar{\chi}} + \partial_\phi\partial_\chi\right) K_\alpha \,, \tag{7.4}$$

$$A^{(1,0)}_{\alpha\beta} = -2\partial_\phi F^+_{\alpha\beta} \,, \tag{7.5}$$

$$G_{\alpha\beta\gamma} = e^{4c_{\alpha\beta\gamma}} \,, \tag{7.6}$$

where the transition functions are chosen to be biholomorphic. Indeed, one can write many more formulas along these lines which would correspond to different but equivalent ways of describing the gerbe with connection. There *always* exists a choice that makes the transition functions (twisted) biholomorphic.

In the case of a holomorphic line bundle the existence of a Hermitian structure (3.8) is equivalent to the existence of a Kähler potential. However, in the context of a holomorphic gerbe, a Hermitian structure only implies the existence of real function $h_{\alpha\beta}$ on double intersections, see (4.2). These functions $h_{\alpha\beta}$ are interpreted as exponents of the Kähler potential for the transition line bundle (see the discussion at the end of section 2). The holomorphic gerbe with Hermitian structure does not naturally produce a real function defined over a patch U_α. However in the present context we can introduce the notion of a bihermitian structure on a (twisted) biholomorphic gerbe that is equivalent to the existence of the generalized Kähler potential.

Suppose that on a bicomplex manifold (M, J_+, J_-) we have transition functions for a biholomorphic gerbe $G_{\alpha\beta\gamma}$ and transition functions for a twisted biholomorphic gerbe $F_{\alpha\beta\gamma}$. The biholomorphic and twisted biholomorphic gerbes are are both hermitian if the following condition is satisfied:

$$G_{\alpha\beta\gamma}F^{-1}_{\alpha\beta\gamma} = h^+_{\alpha\beta}h^+_{\beta\gamma}h^+_{\gamma\alpha}\,, \qquad\qquad G_{\alpha\beta\gamma}\bar{F}_{\alpha\beta\gamma} = h^-_{\alpha\beta}h^-_{\beta\gamma}h^-_{\gamma\alpha}\,, \tag{7.7}$$

where $h^\pm_{\alpha\beta}$ are $J_\pm$-holomorphic functions on double intersections. The products $G_{\alpha\beta\gamma}F_{\alpha\beta\gamma}$ are J_+-holomorphic functions on the triple intersections which satisfy the cocycle condition on the four-fold intersections, since both $G_{\alpha\beta\gamma}$ and $F_{\alpha\beta\gamma}$ satisfy the cocycle conditions independently. Moreover, from (7.7) the product $(GF\bar{G}\bar{F})_{\alpha\beta\gamma}$ is a real trivial cocycle. Therefore $G_{\alpha\beta\gamma}F_{\alpha\beta\gamma}$ can be interpreted as the transition functions for a J_+-holomorphic gerbe with Hermitian structure. Analogously, $G_{\alpha\beta\gamma}\bar{F}^{-1}_{\alpha\beta\gamma}$ can be interpreted as the transition functions for a J_--holomorphic gerbe with Hermitian structure.

Furthermore, (7.7) implies the condition

$$h^+_{\alpha\beta}h^+_{\beta\gamma}h^+_{\gamma\alpha}\bar{h}^-_{\alpha\beta}\bar{h}^-_{\beta\gamma}\bar{h}^-_{\gamma\alpha} = h^-_{\alpha\beta}h^-_{\beta\gamma}h^-_{\gamma\alpha}\bar{h}^+_{\alpha\beta}\bar{h}^+_{\beta\gamma}\bar{h}^+_{\gamma\alpha}\,, \tag{7.8}$$

which implies $h^+(\bar{h}^+)^{-1}(h^-)^{-1}\bar{h}^-$ is a trivial real cocycle:

$$h^+_{\alpha\beta}\bar{h}^-_{\alpha\beta}(h^-_{\alpha\beta})^{-1}(\bar{h}^+_{\alpha\beta})^{-1} = e^{K_\alpha}e^{-K_\beta}\,, \tag{7.9}$$

where the real function K_α is defined on U_α and can be interpreted as a generalized Kähler potential. Thus, on a bihermitian local product space, given a biholomorphic and twisted biholomorphic gerbe satisfying the bihermitian compatibility condition (7.7), one can always construct a generalized Kähler potential.

8 Summary and conclusions

We have discussed aspects of gerbes that arise naturally on generalized Kähler geometries. These geometries are bihermitian, with two complex structures and a related single closed three-form H. It is natural to construct two holomorphic gerbes with the same curvature H. The additional structure of the generalized Kähler geometries allows one to describe them in terms of a generalized Kähler potential (away from irregular points). Using this potential, we showed that the two gerbes fit together into a structure we called the biholomorphic gerbe, whose transition functions can be chosen to be holomorphic with respect to both complex structure (biholomorphic). When the complex structures commute, we were able to explicitly reverse the construction and use the gerbe to construct the generalized Kähler potential. We believe that this should be possible in general, and that the biholomorphic gerbe plays the same role for generalized Kähler geometry as the holomorphic line bundle plays for Kähler geometry.

The generalized (Kähler) potential arises naturally as the superspace Lagrange density in the sigma model approach [6]; a geometric interpretation as a generator of symplectomorphisms related to a choice of polarization was given for the case in which the Poisson structures of the manifold are nondegenerate. Here we see that in the complementary (BiLP) case, when the Poisson structures are maximally degenerate, it has an interpretation as a new kind of bihermitian structure on a biholomorphic gerbe. It would be very interesting to understand the general case, which should combine both perspectives. In particular, the global characterization of the generalized Kähler potential should allow for changes of polarization generated by Legendre transforms, as well as generalized Kähler gauge transformations in the transitions between coordinate patches.

Other extensions of this work that are of immediate interest concern irregular points. Generically, these form (real) co-dimension two loci within the manifold, and can carry nontrivial topology — this corresponds to charge for the gerbe connection.

In the appendix, we give a careful discussion of the generalized Kähler structure on $S^3 \times S^1$ viewed as a BiLP. There are other generalized Kähler structures on $S^3 \times S^1$ that exemplify type change; these will be presented in a separate publication.

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A $S^3 \times S^1$

The manifold $S^3 \times S^1$ is a well known example that illustrates many aspects of the previous discussion. It has a bihermitian structure [29, 30] with a nontrivial biholomorphic gerbe.

The curvature of the gerbe is proportional to the volume form of the S^3 factor; classically, we can normalize it as we choose, but in the quantum theory, the normalization determines the level of the corresponding WZW-model.

The manifold has a Bihermitian Local Product (BiLP) structure, and in adapted complex coordinates ϕ, χ, the metric can be written as:

$$ds^2 = \frac{1}{8\pi}\left[\frac{d\phi d\bar{\phi} + d\chi d\bar{\chi}}{\phi\bar{\phi} + \chi\bar{\chi}}\right] . \tag{A.1}$$

Here the direction corresponding to the homothety (uniform rescaling of ϕ, χ) lies along the S^1 factor, and the S^3 is found by fixing $\phi\bar{\phi} + \chi\bar{\chi}$ to a constant. We compactify the homothetic coordinate to S^1 by the restriction

$$1 \leq \phi\bar{\phi} + \chi\bar{\chi} < e^{4\pi} . \tag{A.2}$$

The metric can be given a more familiar form by introducing real coordinates

$$\phi = \mathrm{e}^{r+i\varphi} \sin\theta , \tag{A.3}$$

$$\chi = \mathrm{e}^{r+i\psi} \cos\theta , \tag{A.4}$$

in which the metric becomes

$$ds^2 = \frac{1}{8\pi}\left(dr^2 + d\theta^2 + \sin^2\theta d\varphi^2 + \cos^2\theta d\psi^2\right) . \tag{A.5}$$

The r direction decouples and corresponds to the S^1. The S^3 is described as an interval $(0 \leq \theta \leq \frac{\pi}{2})$ with a torus fibration over it (the torus being coordinatized by $0 \leq \varphi, \psi \leq 2\pi$). The torus degenerates at the endpoints of the interval.

The metric (A.1) can integrated to give a generalized Kähler potential [29]

$$K_\alpha = \frac{1}{8\pi}\left[\frac{1}{2}\left(\ln\phi\bar{\phi}\right)^2 - \int_1^{\frac{\chi\bar{\chi}}{\phi\bar{\phi}}} dx\, \frac{\ln(1+x)}{x}\right] . \tag{A.6}$$

This is well defined in a patch $\mathcal{U}_\alpha$ where $\phi \neq 0$. When ϕ goes to zero we can make a Kähler gauge transformation and go to a generalized Kähler potential

$$K_\beta = \frac{1}{8\pi}\left[-\frac{1}{2}\left(\ln\chi\bar{\chi}\right)^2 + \int_1^{\frac{\phi\bar{\phi}}{\chi\bar{\chi}}} dx\, \frac{\ln(1+x)}{x}\right] , \tag{A.7}$$

which is well defined in the open set $\mathcal{U}_\beta$ where $\chi \neq 0$. The Kähler gauge transformation taking us between $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ is

$$K_\alpha - K_\beta = \frac{1}{8\pi}\left[\ln(\phi\bar{\phi})\ln(\chi\bar{\chi})\right] , \tag{A.8}$$

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which also immediately tells us that, *cf.* (6.1)

$$F^{+}_{\alpha\beta} = \frac{1}{8\pi} \ln\phi \ln\chi\,, \tag{A.9}$$

$$F^{-}_{\alpha\beta} = \frac{1}{8\pi} \ln\phi \ln\bar{\chi}\;. \tag{A.10}$$

This is enough information to define the gerbe for this generalized Kähler manifold. The simplest definition in this case is in terms of the "transition line bundle" on double overlaps as summarized at the end of section 6. The double overlap $\mathcal{U}_{\alpha\beta} = \mathcal{U}_\alpha \cap \mathcal{U}_\beta$ has the topology of a cylinder (an interval times the S^1 factor) times a torus (constructed from the phases of ϕ and χ). The gerbe is specified by giving a holomorphic line bundle on this set. The first Chern class of this bundle is specified by an integer, which determines the relative factor between the volume form on S^3 and the Chern class of the gerbe (and gives the level of the corresponding WZW model).

We can check that this interpretation makes sense since the information that we have is enough to compute the Kähler form of this line bundle as well as the connection one-form and the transition functions. Using the data we have

$$\omega = 2\partial\bar{\partial}\left(\bar{F}^{-} - F^{-}\right) = \frac{1}{4\pi}\left(\frac{d\chi}{\chi}\wedge\frac{d\bar{\phi}}{\bar{\phi}} - \frac{d\phi}{\phi}\wedge\frac{d\bar{\chi}}{\bar{\chi}}\right)\,, \tag{A.11}$$

$$A^{(1,0)} = \frac{1}{8\pi}\left[\ln\bar{\chi}\,\frac{d\phi}{\phi} - \ln\bar{\phi}\,\frac{d\chi}{\chi}\right]\,, \tag{A.12}$$

which, if we use the real coordinates (A.3) becomes

$$\omega = \frac{1}{2\pi}\left(\frac{dr\wedge d\theta}{\sin\theta\cos\theta} + d\psi\wedge d\varphi\right)\;. \tag{A.13}$$

We can also compute H:

$$H = \frac{1}{\pi}\sin\theta\cos\theta d\theta\wedge d\varphi\wedge d\psi\,, \tag{A.14}$$

which is the volume form of S^3 with the normalization 2π so that $\frac{H}{2\pi}$ is integral.

We could now go on to find the transition functions of the gerbe on triple overlaps. To do this we would first need to subdivide $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ to define a good cover. The nontrivial transition functions are then associated with the phases one picks up going around the torus defined from the phases of ϕ and χ.

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